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Riemann's function for a Klein–Gordon equation with a non-constant coefficient

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Abstract. The Riemann function (or the Riemann–Green function) for the equation $U_{tt} - U_{xx} + (1+\lambda x)U = 0$ is derived by means of a Laplace transform technique. It is shown that the solution given by Whitham's variational method can be identified as the first term in an asymptotic expansion, in λ and $1/t$, of the exact solution.

1. Introduction

A large number of dispersive wave phenomena in various branches of physics are governed by a Klein–Gordon type of equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + U = 0 \tag{1}$$

with the corresponding dispersion relation

$$\omega^2 = 1 + k^2. \tag{2}$$

In electrotechnics the equation is well known as the 'telegraphist's equation', and in microwave theory the dispersion relations for individual wave modes in various wave guides are often of the form (2). In plasma physics the equation frequently appears, describing for instance electromagnetic wave propagation in an unmagnetized plasma, or longitudinal electron plasma waves (Langmuir waves) in a hot plasma.

For a non-uniform medium it often happens that the equation which replaces (1) is

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + (1+\lambda x)U = 0 \tag{3}$$

where λ is a constant, and $1/\lambda$ describes the length scale for the spatial variation of the medium. This is the case for instance for electromagnetic waves in a constant plasma density gradient (Budden 1961, chap. 16) as well as for electron plasma waves in the presence of a uniform static electric field (Wahlberg 1976). The standard technique to investigate such wave problems is to assume harmonic time dependence, i.e.

$$U(x, t) = a(x) \exp(-\mathrm{i}\omega t) \tag{4}$$

in which case the equation, by means of a simple transformation, can be written as the Airy equation (or the Stokes equation; see e.g. Abramowitz and Stegun 1968, p 446). Although one in principle can construct arbitrary wave profiles, such as localized wave packets, by superposing Airy waves of the form (4), this procedure is in practice limited to strictly monochromatic waves, and is not useful to solve e.g. an initial value problem.

In the present paper we present an alternative approach to the problem, based on the construction of the Riemann function (or the Riemann–Green function; cf Courant and Hilbert 1962, p 449; Copson 1958) associated with equation (3). Knowledge of the Riemann function then gives us the possibility to explicitly write down the solution to the general Cauchy problem. In practice, the solution to equation (3), together with initial conditions, then reduces to a problem of evaluating a definite integral.

If the medium is only weakly non-uniform (i.e. $|\lambda| \ll 1$), another possible approach that is not limited to monochromatic waves, is to study the asymptotic ($t \to \infty$) wave field by means of Whitham's variational method (Whitham 1974, chap. 11), or some other similar perturbation technique, such as multiple time scaling. The variational method is also carried out in the paper, and specifically it is shown that the solution that is found can be identified as the first term in an asymptotic expansion (in λ and $1/t$) of the exact solution.

2. The Riemann function

The Cauchy problem of finding a solution to equation (3), with given values of U and its first derivatives on the curve C (figure 1), which has the property that no characteristic ($x \pm t = \text{constant}$) cuts it in more than one point, has a unique solution given by

$$U(x_0, t_0) = \tfrac{1}{2}[UV]_A + \tfrac{1}{2}[UV]_B + \tfrac{1}{2}\int_{AB}\left[\left(V\frac{\partial U}{\partial t} - U\frac{\partial V}{\partial t}\right)\mathrm{d}x + \left(V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right)\mathrm{d}t\right] \tag{5}$$

where A and B are the points in which C is cut by the characteristics through $P(x_0, t_0)$, and $V = V(x, t; x_0, t_0)$, the Riemann function, satisfies the boundary value problem

$$\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} + (1 + \lambda x)V = 0 \tag{6}$$

$$V = 1 \qquad \text{on} \qquad (x - x_0)^2 - (t - t_0)^2 = 0. \tag{7}$$

Copson (1958) gives in a review article a catalogue of methods that have been devised to find the Riemann function for different types of equations. However, the method that will be used in the following is rather different from any of those, although it has some similarities with the Fourier integral method used by Titchmarsh (1937,

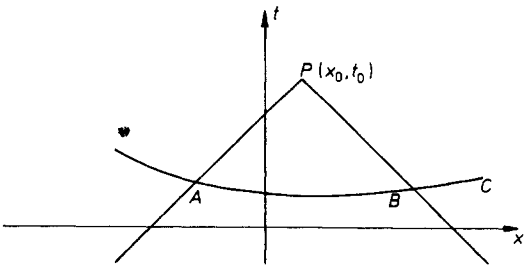

Figure 1. The characteristics of equation (3), passing the point P. C is a continuous curve on which Cauchy data are given.

p 297) to solve (6) and (7) in the special case $\lambda = 0$. We start by introducing the new variables

$$r = \tfrac{1}{2}\lambda^{1/3}[(x_0+t_0)-(x+t)] \qquad s = \tfrac{1}{2}\lambda^{1/3}[(x-t)-(x_0-t_0)]. \tag{8}$$

The problem (6) and (7) then reduces to finding the function $W(r, s)$ such that

$$\frac{\partial^2 W}{\partial r \partial s} + (\sigma + s - r)W = 0 \tag{9}$$

$$W = 1 \qquad \text{when} \qquad rs = 0 \tag{10}$$

where $\sigma = \lambda^{-2/3}(1+\lambda x_0)$. Equation (9) can be solved by use of Laplace transformation. We write the transformation of $W(r, s)$ with respect to s as

$$\tilde{W}(r, p) \equiv \int_0^\infty e^{-ps} W(r, s)\, ds.$$

Then the transformed version of (9) is

$$p\frac{\partial \tilde{W}}{\partial r} - \frac{\partial \tilde{W}}{\partial p} + (\sigma - r)\tilde{W} = 0 \tag{11}$$

where we have made use of the boundary condition (10). Equation (11) is of first order and by elementary means one finds that the general solution is

$$\tilde{W}(r, p) = F_1(r + \tfrac{1}{2}p^2) \exp[(\sigma - r)p - \tfrac{1}{3}p^3].$$

The function F_1 can be determined by the condition $W(0, s) = 1$, i.e. $\tilde{W}(0, p) = 1/p$, which yields

$$F_1(\alpha) = (2\alpha)^{-1/2} \exp[-\sigma(2\alpha)^{1/2} + \tfrac{1}{3}(2\alpha)^{3/2}]$$

and the solution to (9) and (10) is formally given by the inversion integral ($p = \xi + i\eta$)

$$W(r, s) = \frac{1}{2\pi i}\int_{\xi_0 - i\infty}^{\xi_0 + i\infty} (p^2 + 2r)^{-1/2} \exp[(\sigma + s - r)p - \tfrac{1}{3}p^3 - \sigma(p^2+2r)^{1/2} + \tfrac{1}{3}(p^2+2r)^{3/2}]\, dp \tag{12}$$

where $\xi_0 > 0$. If we make a cut in the p plane from $p = -i(2r)^{1/2}$ to $p = i(2r)^{1/2}$ along the $i\eta$ axis, the integrand in (12) becomes analytic in the remainder of the p plane so that (figure 2)

$$\int_\gamma + \sum_{i=1}^{11} \int_{\gamma_i} = 0.$$

It is straightforward to show that (cf Doetsch 1970, p 174):

$$\lim_{R\to\infty} \int_{\gamma_{1,11}} = 0, \qquad \lim_{R\to\infty} \int_{\gamma_{2,10}} = 0, \qquad \lim_{\epsilon\to 0} \int_{\gamma_{5,7}} = 0.$$

Thus

$$W(r, s) = \frac{1}{\pi}\int_{-(2r)^{1/2}}^{(2r)^{1/2}} (2r - \eta^2)^{-1/2} \cosh[\sigma(2r-\eta^2)^{1/2} - \tfrac{1}{3}(2r-\eta^2)^{3/2}] \times \cos[(\sigma + s - r)\eta + \tfrac{1}{3}\eta^3]\, d\eta. \tag{13}$$

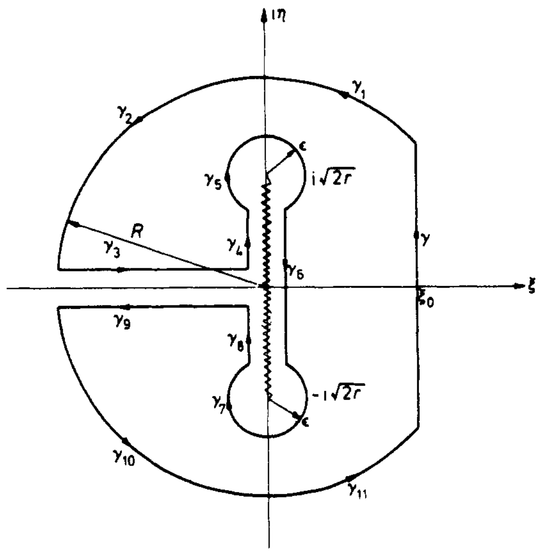

Figure 2. Contour of integration for the Laplace inversion.

By means of the substitution $\sin\phi = \eta/(2r)^{1/2}$ and by using symmetry properties of the integrand, it is easily verified that (13) can be written

$$W(r,s)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos[s(2r)^{1/2}a(\phi)+\sigma(2r)^{1/2}b(\phi)+(2r)^{3/2}c(\phi)]\,\mathrm{d}\phi \quad (14)$$

where

$$\begin{aligned} a(\phi)&=\sin\phi\\ b(\phi)&=\sin\phi+\mathrm{i}\cos\phi \quad (15)\\ c(\phi)&=\tfrac{1}{3}\sin^3\phi-\tfrac{1}{2}\sin\phi-\mathrm{i}\tfrac{1}{3}\cos^3\phi. \end{aligned}$$

We can extract a useful property of $W(r,s)$, as given by (14), from the following consideration. By taking derivatives with respect to x, y and z of the function

$$f(x,y,z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos[xya(\phi)+zxb(\phi)+x^3c(\phi)]\,\mathrm{d}\phi$$

and observing that, from (15)

$$xya'(\phi)+zxb'(\phi)+x^3c'(\phi)$$
$$=-\mathrm{i}[xya(\phi)+zxb(\phi)+3x^3c(\phi)-2xya(\phi)+xyb(\phi)+\tfrac{1}{2}x^3b(\phi)]$$

it is seen that

$$x\frac{\partial f}{\partial x}-2y\frac{\partial f}{\partial y}+\left(y+\frac{1}{2}x^2\right)\frac{\partial f}{\partial z}=\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}}{\mathrm{d}\phi}\{\cos[xya(\phi)+zxb(\phi)+x^3c(\phi)]\}\,\mathrm{d}\phi=0. \quad (16)$$

Again we have an equation of first order, and it is easily shown that any solution to (16) must be a function of the form

$$f(x,y,z)=F_2(z-\tfrac{1}{4}x^2+\tfrac{1}{2}y,\,x\sqrt{y}).$$

Thus $W(r, s)$, as given by (14), must be a function of the two variables

$$u = rs \qquad v = \sigma + \tfrac{1}{2}s - \tfrac{1}{2}r. \tag{17}$$

This simply means that if we want to calculate $W(r, s)$ for some values of (r, s, σ) we can choose a new set of values (r', s', σ') and get the same result as long as

$$r's' = rs \qquad \sigma' + \tfrac{1}{2}(s' - r') = \sigma + \tfrac{1}{2}(s - r).$$

Let us then explicitly take $(2r')^{1/2} = \rho$, i.e. $r' = \tfrac{1}{2}\rho^2$, in which case

$$s' = 2u/\rho^2 \qquad \sigma' = v - u/\rho^2 + \tfrac{1}{4}\rho^2.$$

Then

$$(\sigma' + s' - r')(2r')^{1/2} = v\rho + u/\rho - \tfrac{1}{4}\rho^3$$

$$\sigma'(2r')^{1/2} = v\rho - u/\rho + \tfrac{1}{4}\rho^3$$

and we see that

$$W(r, s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\{v\rho(\sin\phi + \mathrm{i}\cos\phi) + u/[\rho(\sin\phi + \mathrm{i}\cos\phi)]$$
$$- \tfrac{1}{12}\rho^3(\sin 3\phi + \mathrm{i}\cos 3\phi)\}\, \mathrm{d}\phi.$$

However, this is the contour integral

$$W(r, s) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \cos\left(vz + \frac{u}{z} + \frac{1}{12}z^3\right)\frac{\mathrm{d}z}{z} \tag{18}$$

where Γ is defined by $|z| = \rho$, encircling the origin in the positive direction. Since the integrand of (18) is analytic everywhere except at $z = 0$, Γ can be taken to be any closed contour encircling the origin in the positive direction. If we further make use of the formula $2\cos z = \exp(\mathrm{i}z) + \exp(-\mathrm{i}z)$, (18) can alternatively be written

$$W(r, s) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \exp\left(vz - \frac{u}{z} - \frac{1}{12}z^3\right)\frac{\mathrm{d}z}{z}. \tag{19}$$

It is easy to see that (19), with u and v given by (17), indeed satisfies (9) and (10). First of all it is immediately seen that (10) is satisfied because of the residue theorem. Secondly, with $W(r, s)$ given by (19), we have

$$\frac{\partial^2 W}{\partial r\,\partial s} + (\sigma + s - r)W = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\mathrm{d}}{\mathrm{d}z}\left[\exp\left(vz - \frac{u}{z} - \frac{1}{12}z^3\right)\frac{1}{z}\right]\mathrm{d}z = 0.$$

If we go back to the original (x, t) coordinates we find that the Riemann function is given by

$$V(x, t; x_0, t_0) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \exp\{[1 + \tfrac{1}{2}\lambda(x + x_0)]z - [(t - t_0)^2 - (x - x_0)^2]/4z - \tfrac{1}{12}\lambda^2 z^3\}\frac{\mathrm{d}z}{z}. \tag{20}$$

For numerical calculation it is useful to expand the factor $\exp(-\lambda^2 z^3/12)$ of the integrand in a Taylor series, and make use of the formula (Ince 1956, p 466)

$$J_n(\alpha) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma z^{-n-1}\exp\left[\frac{1}{2}\alpha\left(z - \frac{1}{z}\right)\right]\mathrm{d}z$$

for the Bessel function of order n. This yields

$$V(x, t; x_0, t_0) = \sum_{n=0}^{\infty} \frac{\Omega^n}{n!} J_{3n}(\Lambda) \tag{21}$$

where

$$\Omega = \frac{\lambda^2}{96}\left(\frac{(t-t_0)^2-(x-x_0)^2}{1+\frac{1}{2}\lambda(x+x_0)}\right)^{3/2}$$

$$\Lambda = \{[1+\tfrac{1}{2}\lambda(x+x_0)][(t-t_0)^2-(x-x_0)^2]\}^{1/2}.$$

Specifically we see that when $\lambda = 0$ only the first term in the series (21) is different from zero, and we get the well known result (Courant and Hilbert 1962, p 455; Copson 1958, p 331, p 347; Titchmarsh 1937, p 298)

$$V_{\lambda=0} = J_0\{[(t-t_0)^2-(x-x_0)^2]^{1/2}\}.$$

Moreover, if λ is non-zero but small, i.e. if the medium is weakly non-uniform, we have $\Omega \ll 1$, and (21) is rapidly converging.

3. The variational method

For a weakly non-uniform medium, i.e. $|\lambda| \ll 1$ in equation (3), Whitham's variational method (Whitham 1974, chap. 11) may be used in order to determine the structure of the asymptotic wave field. Indeed, it is immediately seen that the Euler equation, corresponding to the variational problem

$$\delta \int L\left(\frac{\partial U}{\partial t}, \frac{\partial U}{\partial x}, U\right) \mathrm{d}x\, \mathrm{d}t = 0$$

with the Lagrangian

$$L = \frac{1}{2}\left(\frac{\partial U}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2 - \frac{1}{2}(1+\lambda x)U^2$$

is equation (3). Due to the dispersive character of the medium, we expect that the asymptotic wave field will have the form of a slowly varying non-uniform wave train

$$U(x,t) \sim a(x,t)\cos(\theta(x,t)+\eta). \tag{22}$$

This is the basic postulate of Whitham's theory. The method, given by Whitham, in order to find the equations for the amplitude a and the phase θ, is to formulate the 'average variational principle'

$$\delta \int \bar{L}\left(-\frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x}, a\right) \mathrm{d}x\, \mathrm{d}t = 0 \tag{23}$$

where $\bar{L}$, the 'averaged Lagrangian', is defined by

$$\bar{L}\left(-\frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x}, a\right) \equiv \frac{1}{2\pi}\int_0^{2\pi} L\, \mathrm{d}\theta$$

and θ_t, θ_x and a are kept constant in the averaging process. In our case we find

$$\bar{L} = \tfrac{1}{4}a^2\omega^2 - \tfrac{1}{4}a^2k^2 - \tfrac{1}{4}(1+\lambda x)a^2$$

where local frequency and wave number have been defined by

$$\omega \equiv -\partial\theta/\partial t \qquad k \equiv \partial\theta/\partial x. \tag{24}$$

In the general case, the Euler equations resulting from (23) are (Whitham 1974, p 393)

$$\delta a : \frac{\partial \bar{L}}{\partial a} = 0 \qquad \delta\theta : \frac{\partial}{\partial t}\left(\frac{\partial \bar{L}}{\partial \omega}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \bar{L}}{\partial k}\right) = 0.$$

Moreover, since we prefer to work with the variables ω and k, instead of θ, the regularity condition $\theta_{xt} = \theta_{tx}$ must be added as a third equation. In our case this yields

$$\omega^2 = 1 + \lambda x + k^2 \tag{25a}$$

$$\frac{\partial}{\partial t}(a^2\omega) + \frac{\partial}{\partial x}(a^2 k) = 0 \tag{25b}$$

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{25c}$$

The first of these equations is the local dispersion relation, and together with the third equation it determines the geometry of the wave pattern. The second equation is the equation for amplitude, and can be interpreted as a conservation equation for 'wave action', i.e. energy density divided by local frequency. In fact, it is directly seen that (25*b*) can be written

$$\frac{\partial}{\partial t}\left(\frac{\bar{W}}{\omega}\right) + \frac{\partial}{\partial x}\left(v_{gr}\frac{\bar{W}}{\omega}\right) = 0 \tag{26}$$

where $\bar{W}$ is the energy density

$$\bar{W} = \omega \frac{\partial \bar{L}}{\partial \omega} \tag{27}$$

and v_{gr} the group velocity

$$v_{gr} = \frac{\partial \omega}{\partial k}. \tag{28}$$

Now, let us determine the structure of the asymptotic wave field (22) resulting from an initial ($t = 0$) perturbation around $x = 0$. With the dispersion relation (25*a*) substituted into (25*c*), the latter equation can be looked upon as a hyperbolic wave equation for the wave number $k(x, t)$. The velocity of propagation is the group velocity (28) and the characteristics (group lines) of the equation are given by

$$\frac{dx}{dt} = v_{gr}. \tag{29}$$

Moreover, it is generally true for time-independent media that the frequency remains constant along the group lines (Whitham 1974, p 383). The dispersion relation (25*a*) then shows that the wave number varies according to

$$k(x) = \pm(\omega_0^2 - 1 - \lambda x)^{1/2}$$

on the group line along which the frequency ω_0 is 'propagating'. Then it is straightforward to integrate (29) to find that the group lines are explicitly given by

$$t(\omega_0, x) = \pm 2\omega_0\lambda^{-1}[(\omega_0^2-1)^{1/2} - (\omega_0^2 - 1 - \lambda x)^{1/2}]. \tag{30}$$

It is not possible to calculate the amplitude of the entire asymptotic wave field until the initial conditions are further specified, but the amplitude variation along the different group lines can be found from the following argument. Consider two adjacent group lines (ω_0, $\omega_0 + \mathrm{d}\omega_0$) emanating from $(x, t) = (0, 0)$. From (26) it follows that the amount of wave action between the group lines, i.e. $\bar{W}\,\mathrm{d}x/\omega_0$, where $\mathrm{d}x$ is the distance between the group lines, is conserved. Thus

$$\bar{W} \sim \left|\frac{\mathrm{d}\omega_0}{\mathrm{d}x}\right|. \tag{31}$$

But

$$t(\omega_0 + \mathrm{d}\omega_0, x + \mathrm{d}x) \simeq t(\omega_0, x) + \mathrm{d}\omega_0 \frac{\partial t}{\partial \omega_0} + \mathrm{d}x \frac{\partial t}{\partial x}$$

so that, by virtue of (29) and (31)

$$\bar{W} \sim \left| v_{\mathrm{gr}}(\omega_0, x) \frac{\partial}{\partial \omega_0} t(\omega_0, x) \right|^{-1}.$$

In our case, if we make use of the formulae (27), (28) and (30), we find that the amplitude variation along the group lines is

$$a \sim t^{-1/2}(\omega_0, x)\left|\omega_0^2 - (\omega_0^2 - 1)^{1/2}(\omega_0^2 - 1 - \lambda x)^{1/2}\right|^{-1/2}. \tag{32}$$

4. Asymptotic behaviour of the exact solution

In this section we show that the results given above in § 3 are contained in the first term in an asymptotic expansion of the exact solution (5), $V(x, t; x_0, t_0)$ given by (20). In principle, as we know the Riemann function, we could calculate $U(x, t)$ resulting from any specific choice of initial conditions. However, let us for simplicity take

$$U(x, 0) = \delta(x) \qquad \frac{\partial U}{\partial t}(x, 0) = 0.$$

The δ-pulse is a convenient choice of the initial disturbance because it represents the entire wavelength spectrum, and also because it considerably simplifies the evaluation of the integral in (5). In fact, it is immediately seen that the first two terms in (5) in this case give U at the wave fronts, so that, except for wave front behaviour, the wave field is simply

$$U(x_0, t_0) = -\frac{1}{2}\frac{\partial V}{\partial t}(0, 0; x_0, t_0)$$

or, by virtue of (20) ($x_0 \to x$, $t_0 \to t$)

$$U(x, t) = -\frac{t}{8\pi\mathrm{i}} \int_\Gamma \exp\left[\left(1 + \frac{1}{2}\lambda x\right)z - \frac{t^2 - x^2}{4z} - \frac{1}{12}\lambda^2 z^3\right]\frac{\mathrm{d}z}{z^2}. \tag{33}$$

In order to achieve the desired asymptotic limit of (33) we must first of all let $\lambda \to 0$, corresponding to an infinitely slow spatial variation of the medium, but we must at the same time let $t \to \infty$ and keep $|x/t| < 1$. One realizes how this can be done if we make the substitution

$$\zeta = z\lambda[2(1+\tfrac{1}{2}\lambda x)]^{-1/2}.$$

Then (33) can be written

$$U(x,t) = -\lambda t[32(1+\tfrac{1}{2}\lambda x)]^{-1/2} F(\tau;\kappa)$$

where ($\zeta \to z$)

$$F(\tau;\kappa) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \exp \tau\left(z - \frac{\kappa}{2z} - \frac{1}{6}z^3\right)\frac{\mathrm{d}z}{z^2} \tag{34}$$

and

$$\tau = \frac{2^{1/2}(1+\frac{1}{2}\lambda x)^{3/2}}{\lambda} \qquad \kappa = \frac{(\lambda t)^2(1-x^2/t^2)}{4(1+\frac{1}{2}\lambda x)^2}.$$

In the following it is necessary to restrict the consideration to the range $x > -1/\lambda$ (λ assumed positive), where the local dispersion relation (25*a*) is formally similar to (2). Then, a description of the entire wave field can be given up to $t = 1/\lambda$, which means that we must also have $x < 1/\lambda$. Since $x^2 < t^2$, it is seen that κ must be in the range

$$0 < \kappa \leqslant 1. \tag{35}$$

Thus the desired asymptotic limit is achieved by letting $\tau \to \infty$, in which case (34) is readily estimated by straightforward application of the saddle point method. Defining

$$h(z) \equiv z - \frac{\kappa}{2z} - \frac{1}{6}z^3$$

the saddle points are given by the roots of the equation $h'(z) = 0$ (see e.g. de Bruijn 1958, chap. 5), i.e.

$$z_{1,2} = \pm(\mu+1)^{1/2} \qquad z_{3,4} = \pm \mathrm{i}(\mu-1)^{1/2} \tag{36}$$

where $\mu = (\kappa+1)^{1/2}$. Thus the contour Γ in (34) should pass through a convenient number of four points (36). Moreover, the contour must pass the saddle points at an angle $\chi_k \pm \pi$, where

$$\chi_k = \tfrac{1}{2}\pi - \tfrac{1}{2}\arg h''(z_k).$$

For the points (36) we find

$$\chi_1 = 0, \qquad \chi_2 = \tfrac{1}{2}\pi, \qquad \chi_3 = \tfrac{3}{4}\pi, \qquad \chi_4 = \tfrac{1}{4}\pi.$$

Obviously the contour must then be chosen such as shown in figure 3. The contribution to the integral from each of the three points $z_k = z_2, z_3, z_4$ is (de Bruijn 1958, p 88)

$$\frac{1}{2\pi \mathrm{i}}\left(\frac{2\pi}{|\tau h''(z_k)|}\right)^{1/2} \exp(\mathrm{i}\chi_k + \tau h(z_k))\frac{1}{z_k^2}[1 + \mathrm{O}(1/\tau)]. \tag{37}$$

However, it is straightforward to show that $h(z_2) < 0$ for κ in the range (35), whereas $h(z_3)$ and $h(z_4)$ are purely imaginary, so that only the contributions from z_3 and z_4 are

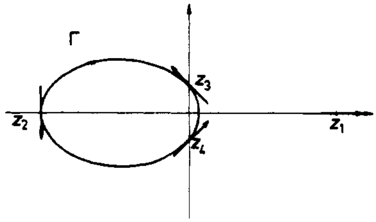

Figure 3. Contour of integration for application of the saddle point method.

of importance. Adding (37) for $z_k = z_3$, z_4 then gives the final result

$$F(\tau;\kappa)=[\pi\tau(\mu-1)^{1/2}(\mu^2-\mu)]^{-1/2}\cos[\tfrac{2}{3}(\mu-1)^{1/2}(\mu+2)\tau-\tfrac{3}{4}\pi]+O(\tau^{-3/2}).$$

Then, after some algebra, and by introducing the functions

$$\psi_1(x,t)=(4+4\lambda x+\lambda^2t^2)^{1/2}$$
$$\psi_2(x,t)=(\psi_1-2-\lambda x)^{1/2}$$
$$\psi_3(x,t)=\psi_1+4+2\lambda x$$

and the amplitude and phase

$$a(x,t)=(8\pi\psi_1 t)^{-1/2}(\lambda t/\psi_2)^{3/2} \tag{38a}$$
$$\theta(x,t)=\psi_2\psi_3/3\lambda \tag{38b}$$

one finds that

$$U(x,t)=a(x,t)\cos(\theta(x,t)+\tfrac{1}{4}\pi)+O(\lambda^{3/2}). \tag{39}$$

Hence we have an oscillatory non-uniform wave train of a form that was expected, and postulated, in the previous section (22). In figure 4 the frequency, wave number (defined by (24)) and amplitude for the wave train (39) is shown at the time $t=1/\lambda$.

Let us see whether this result is consistent with the results found in § 3. Differentiating (38b) with respect to x and t gives, respectively

$$\frac{\partial\theta}{\partial x}=\frac{\psi_1-2-2\lambda x}{2\psi_2}\qquad \frac{\partial\theta}{\partial t}=\frac{\lambda t}{2\psi_2}. \tag{40}$$

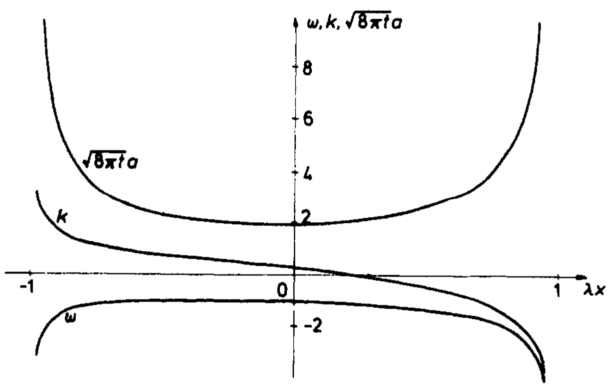

Figure 4. Frequency, wave number and amplitude for the wave train (39).

A straightforward calculation then shows that

$$\left(\frac{\partial\theta}{\partial t}\right)^2-\left(\frac{\partial\theta}{\partial x}\right)^2=1+\lambda x$$

i.e. the wave train (39) does indeed satisfy the previously proposed dispersion relation (25*a*), with ω and k defined by (24). The curves in the (x, t)-plane on which the 'frequency' (40) is constant, are given by

$$\omega_0=-\lambda t/2\psi_2. \tag{41}$$

(Obviously the definition (24) in this case always gives a negative value for ω.) However, it is not difficult to show that equation (41) alternatively can be written in the form (30), i.e. the expression for the group lines. Finally, the amplitude along these lines is, by virtue of (38*a*) and (41), proportional to $(t\psi_1)^{-1/2}$. One can show that along the group lines (30)

$$\psi_1\sim\left|\omega_0^2-(\omega_0^2-1)^{1/2}(\omega_0^2-1-\lambda x)^{1/2}\right|$$

which verifies the previously given formula (32) for the amplitude variation.

5. Conclusions

In this paper we have shown that the Riemann function associated with the Klein–Gordon equation (3) can be derived by means of a Laplace transform technique. The final result may be represented either as a contour integral (20) or as an infinite series of Bessel functions (21). Knowing the Riemann function, the solution of a general Cauchy problem reduces to a problem of evaluating a definite integral, and Riemann's method may be useful in the study of non-monochromatic wave propagation in non-uniform media. If the medium is only weakly non-uniform, however, the asymptotic wave field can be determined by means of a suitable perturbation approach, e.g. Whitham's variational method. In the paper it is also shown that the result of such an approach coincides with the first term in an asymptotic expansion of the exact solution.

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